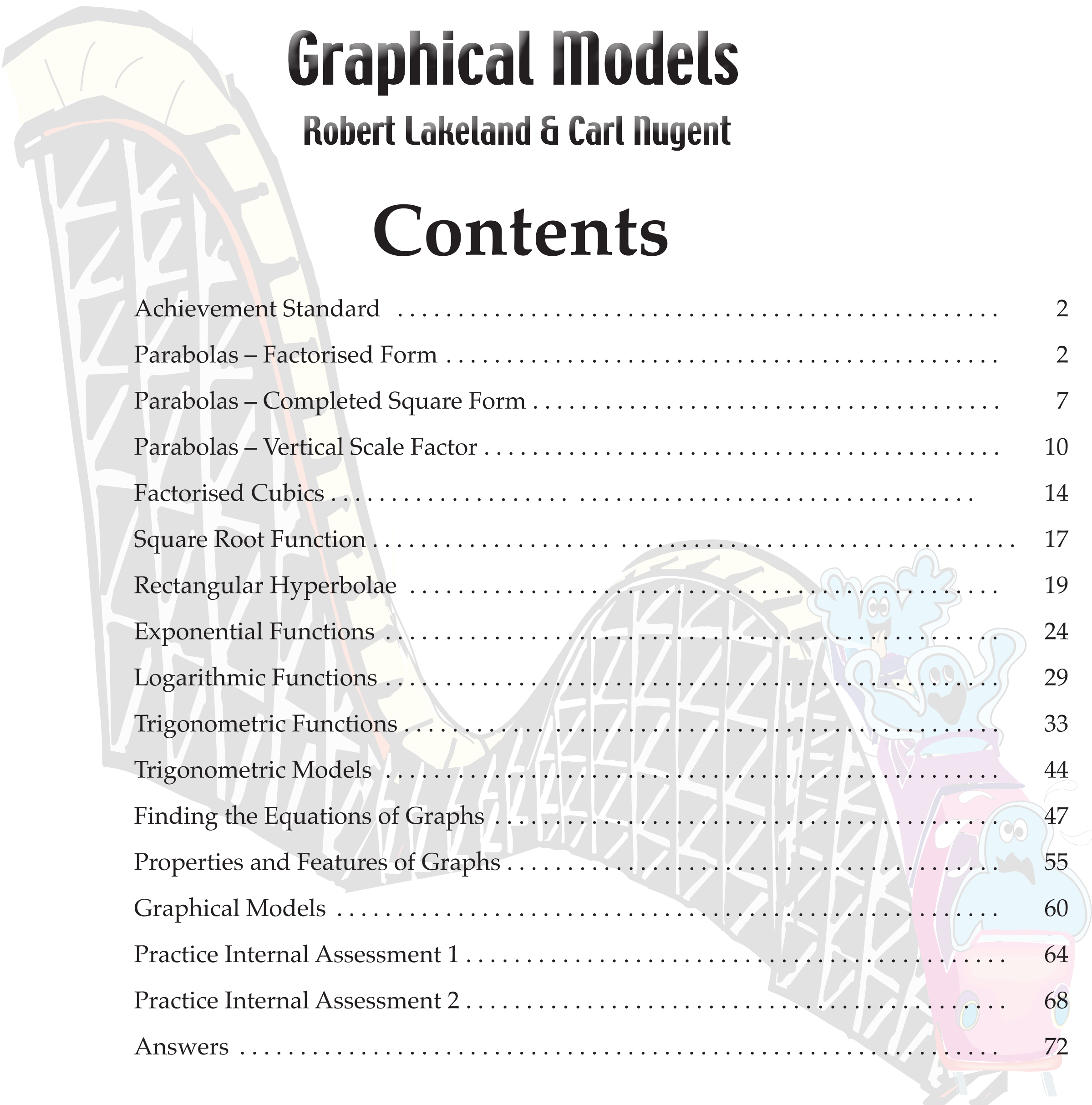
Year 12 Mathematics

IAS 2.2

Graphical Models

Robert Lakeland & Carl Nugent



Contents

nuLake Ltd
Innovative Publisher of Mathematics Texts

NCEA 2 Internal Achievement Standard 2.2 – Graphical Models

This achievement standard involves applying graphical methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply graphical models in solving problems.	• Apply graphical models, using relational thinking, in solving problems.	• Apply graphical models, using extended abstract thinking, in solving problems.

◆ This achievement standard is derived from Level 7 of The New Zealand Curriculum and is related to the achievement objectives
- ❖ display the graphs of linear and non-linear functions and connect the structure of the functions with their graphs
- ❖ form and use linear, quadratic, and simple trigonometric equations

in the Mathematics strand of the Mathematics and Statistics Learning Area.

◆ Apply graphical methods in solving problems involves:
- ❖ selecting and using methods
- ❖ demonstrating knowledge of the properties of functions and graphs
- ❖ communicating using appropriate representations.

◆ Relational thinking involves one or more of:
- ❖ selecting and carrying out a logical sequence of steps
- ❖ connecting different concepts or representations
- ❖ demonstrating understanding of concepts
- ❖ forming and using a model;

and also relating findings to a context, or communicating thinking using appropriate mathematical statements.

◆ Extended abstract thinking involves one or more of:
- ❖ devising a strategy to investigate a situation
- ❖ identifying relevant concepts in context
- ❖ developing a chain of logical reasoning, or proof
- ❖ forming a generalisation;

and also using correct mathematical statements, or communicating mathematical insight.

◆ Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

◆ Methods include a selection from those related to:
- ❖ graphs at curriculum Level 7, their features and their equations
- ❖ transformations of graphs
- ❖ connecting different representations of relations
- ❖ properties of functions (may include domain and range).

Parabolas – Factorised Form

Factorised Parabolas

A simple quadratic function is of the form

$$y = x^2 + bx + c$$

or $$y = (x - d)(x - e)$$

or $$y = \pm(x - a)^2 + b$$

Extra Notes

Parabolas – Vertical Scale Factor

Parabolas – Vertical Scale Factor

If there is a coefficient for the x^2 term such as

$$y = k(x + c)(x + d)$$

$$y = k(x - a)^2 + b$$

then the resulting parabola is stretched vertically by a scale factor k.

Scale Factor in Factorised Form

If the quadratic is factorised (i.e. written as a product of factors) then we know the x intercepts.

Consider $y = 2(x - 1)(x + 3)$

The x intercepts are still (1, 0) and $(^-3, 0)$ because when a factor is zero (e.g. when x = 1) the coefficient has no effect.

The coefficient, however, does affect the y intercept and turning point.

To find the y intercept (i.e. where the graph cuts the y axis) we substitute x = 0 into the equation.

$y = 2(0 - 1)(0 + 3)$

$y = 2 \times {^-1} \times 3$

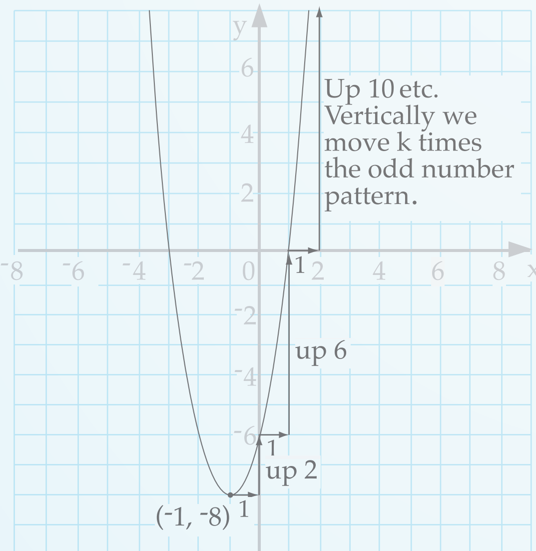
$y = {^-6}$ y intercept $(0, {^-6})$

To find the turning point we use the midpoint between the x intercepts (i.e. $x = {^-1}$). Substituting this into the equation gives

$y = 2({^-1} - 1)({^-1} + 3)$

$y = 2 \times {^-2} \times 2$

$y = {^-8}$ Turning point $({^-1}, {^-8})$



Scale Factor in Completed Square Form

If the quadratic is in completed square form (i.e. written as $y = k(x - a)^2 + b$) then we know the turning point (a, b).

Consider $y = 3(x + 1)^2 - 4$

The base or turning point is at $({^-1}, {^-4})$.

From the base we normally move out 1 up 1, out 1 up 3, out 1 up 5, out 1 up 7 etc.

With a coefficient of x^2 of k we move out 1 up 1k, out 1 up 3k, out 1 up 5k, out 1 up 7k etc.

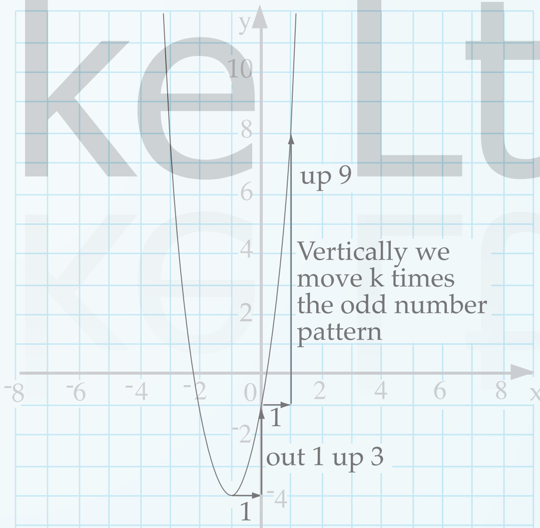
In the case of $y = 3(x + 1)^2 - 4$ we move out 1 up 1 x 3, out 1 up 3 x 3, out 1 up 3 x 5, out 1 up 3 x 7 and so on.

We still substitute x = 0 into the equation to find the y intercept.

$y = 3(0 + 1)^2 - 4$

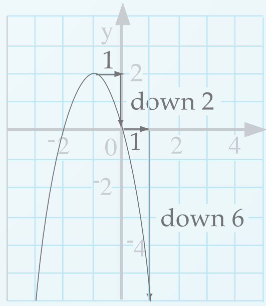
$y = 3 \times 1 - 4$

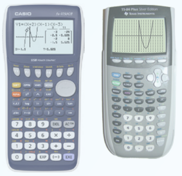
$y = {^-1}$ y intercept $(0, {^-1})$



When the coefficient of x^2 or vertical scale factor is negative, the parabola will be inverted (upside down).

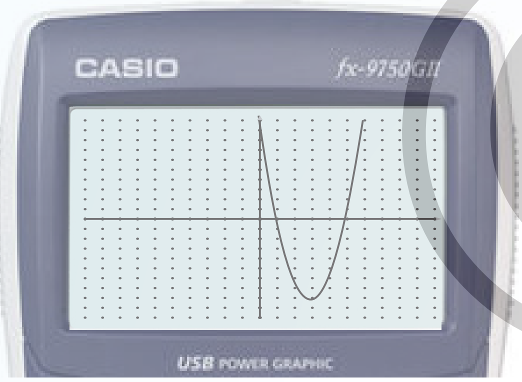
For example, with $y = {^-2}(x + 1)^2 + 2$ the base or turning point is at the position $({^-1}, 2)$ but from this position we move out 1 down 2 x 1, out 1 down 2 x 3, out 1 down 2 x 5 etc.





If you graph a parabola with your graphics calculator, you can use the Graph Solve feature to calculate the x intercepts (called roots or zero) and the coordinates of any turning point (either a maximum or minimum point). For example, to find the minimum point for $y = 2(x-5)(x-1)$:

On the Casio 9750GII graph the parabola so you can see all the intercepts and the turning point. If you cannot see a point the calculator will not solve for it. You will then have to adjust the view window so all relevant points are displayed.



The turning point.

Select the graph solve menu and the minimum (MIN) [or maximum if it is upside down].

	G-Solv	MIN
SHIFT	F5	F3

The calculator makes a good estimate of the minimum but sometimes you may need to round the answer appropriately.

With $y = 2(x-5)(x-1)$ the answer is

$x = 3$ and $y = {}^-8$, i.e. $(3, {}^-8)$.

The intercepts

The y intercept is found from the graph solve menu similarly to the minimum point.

	G-Solv	Y-ICPT
SHIFT	F5	F4

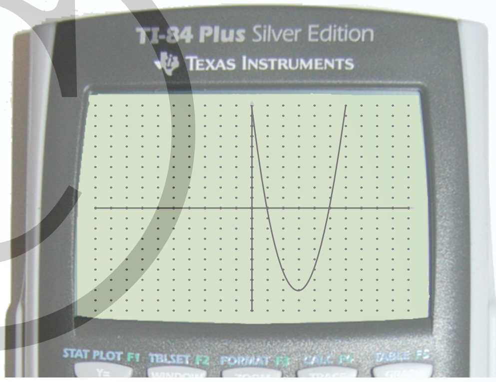
With $y = 2(x-5)(x-1)$ the answer is (0, 10).

The x intercept is called a root. There are usually two roots and the calculator will only find one at a time then you use the right arrow to find the second root.

	G-Solv	ROOT
SHIFT	F5	F1

returns an answer of (1, 0) for $y = 2(x-5)(x-1)$. Selecting the right arrow then gives the second root (5, 0).

On the TI-84 Plus graph the parabola making sure you can see all the intercepts and the turning point. If you cannot see a point the calculator will not solve for it. You will then have to adjust the window so all relevant points are displayed.



The turning point.

Select the calculate menu and the minimum (MIN) [or maximum if it is upside down].

	CALC	Min.
2nd	TRACE	3

You will need to use the arrow keys and ENTER to select the region (left side and right side) for the calculator search.

The calculator makes a good estimate of the minimum but you may need to round the answer appropriately. With $y = 2(x-5)(x-1)$ the answer is

$x = 2.999\,999\,3$ and $y = {}^-8$

which you will round to $(3, {}^-8)$.

The intercepts

The y intercept is found by evaluating the graph at $x = 0$.

	CALC	Value	X =	
2nd	TRACE	1	0	ENTER

With $y = 2(x-5)(x-1)$ the answer is (0, 10).

The x intercept is called a zero. There are usually two roots and again you will need to select the region to search using the arrow keys and ENTER for each root independently.

In the case of $y = 2(x-5)(x-1)$ the calculator correctly returns the x intercepts of (1, 0) and (5, 0).

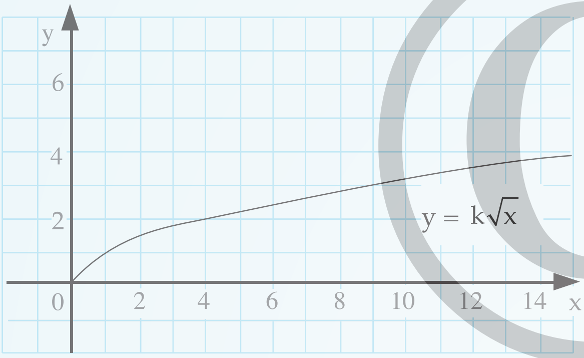
Square Root Function

Square Root Function

Another function we can use to model with is the square root function

i.e. $y = k\sqrt{x}$

where 'k' is a scale factor that stretches the function parallel to the y axis depending on how far the function is from the x axes.



Like all functions, the graph can be translated from the origin. Consider

$y = f(x)$

The function $y = f(x - a) + b$

OR $y - b = f(x - a)$

is a translation of f(x) across a and up b.

Therefore the square root function

$$y = k\sqrt{(x-a)} + b$$

is the square root function starting from the point (a, b).

The square root function

$$y = 2\sqrt{(x-3)} + 1$$

starts from (3, 1) and the function is stretched parallel to the x axes by a scale factor of 2. We can sketch using these transformations or by plotting points.

Use the transformation of the function y = f(x).

If we know $y = f(x)$

Then $y = f(x - a) + b$

OR $y - b = f(x - a)$

is the same function translated as though the origin is at the point (a, b).

If f(x) represents the stick figures on the right then y = f(x + 5) + 2 represents the translated (back 5, up 2) figure.

Extra Notes

Example

Graph the function

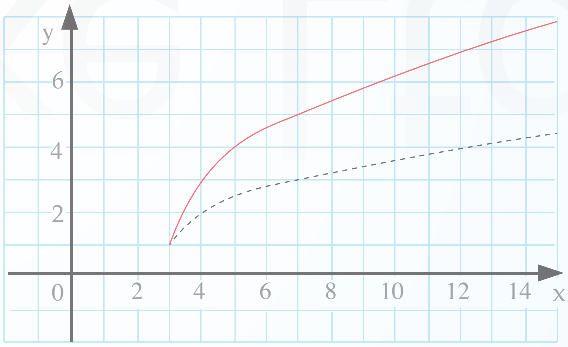
$$y = 2\sqrt{(x-3)} + 1$$

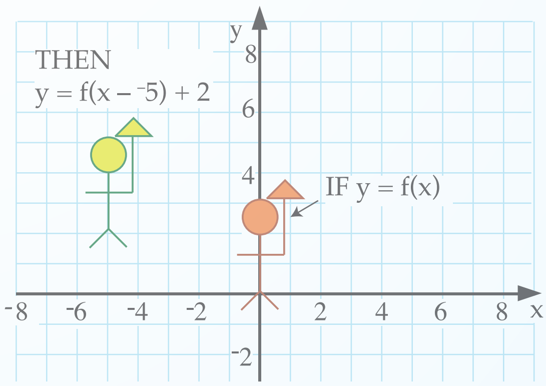
We could generate a table starting with x = 3. To get the values in the right hand column we substitute our chosen x values.

The Table function on your graphics calculator will also produce this table.

x value	$y = 2\sqrt{(x-3)} + 1$
3	1
4	3.00
5	3.83
6	4.46
7	5.00
8	5.47
9	5.90
10	6.29
11	6.66

Alternatively, we could sketch from the translated origin (3, 1) then stretch it so all points are twice as far from the base line y = 1.





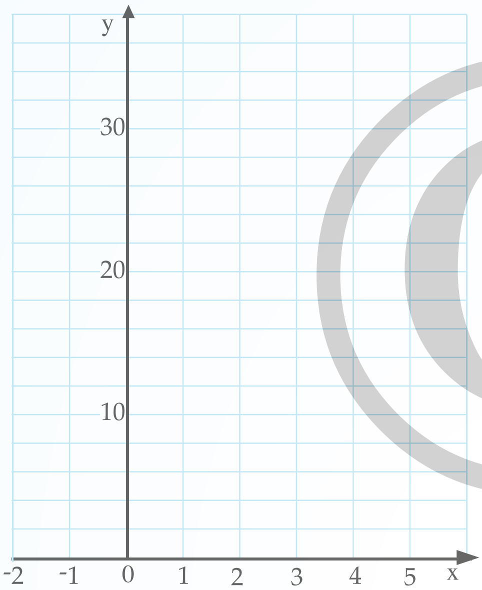
Achievement – Graph the exponential functions of the form $y = a^x$, identifying the given point on each graph.

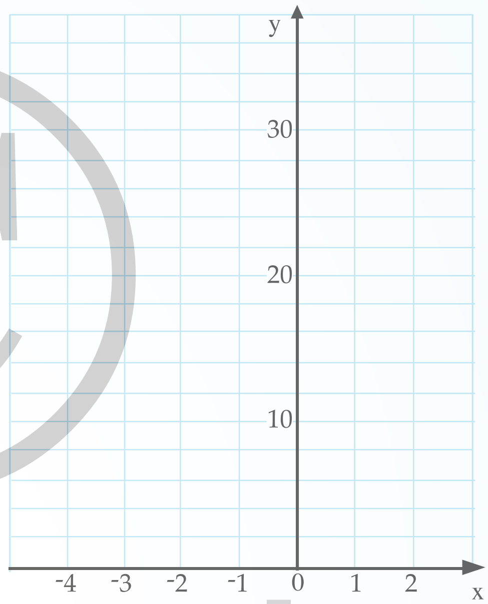
51. $y = 3^x$ identifying $x = 2$.

53. $y = 2^{-x}$ identifying $x = {}^-2$.

52. $y = 4^x$ identifying $x = 2$.

54. $y = 2^{x+3}$ identifying $x = {}^-2$.





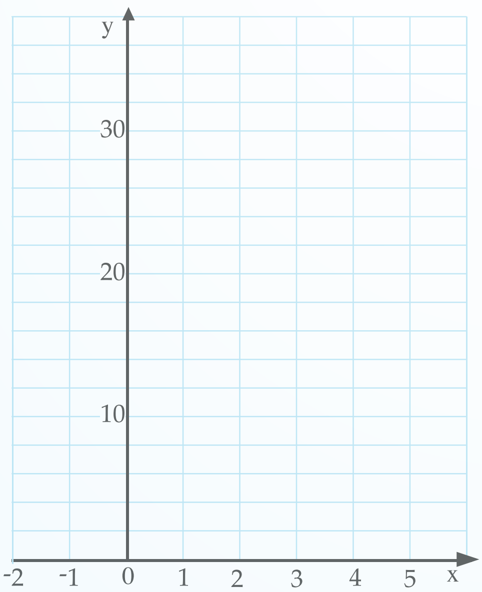
Graph the exponential functions and describe the transformation of $y = a^x$ that would map it onto your graph.

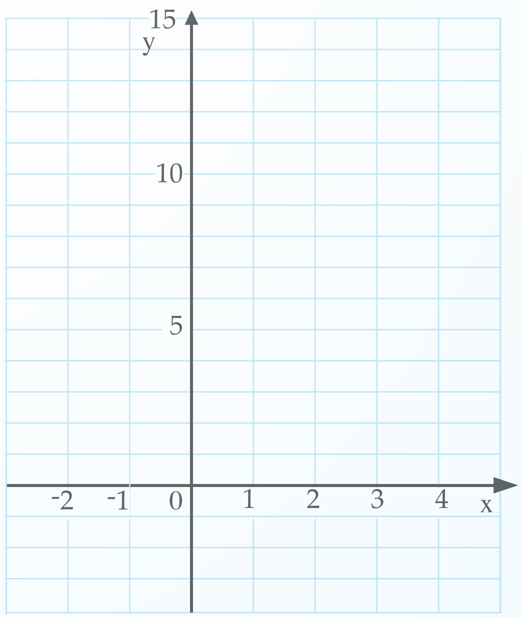
55. $y = 3^x + 6$ involving translation.

57. $y = 4 - 2^x$ involving reflection.

56. $y = 4^{x-2}$ involving translation.

58. $y = 4^{(2-x)}$ involving reflection.

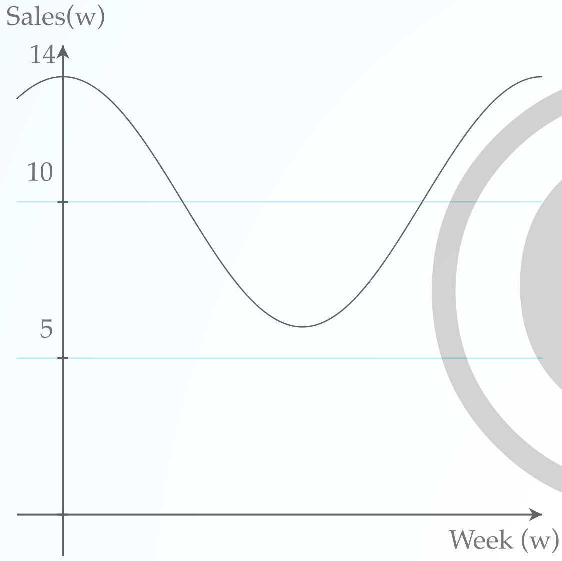




120. The sales that a real estate firm makes over a 12 month period are being modelled by the equation

$$\text{Sales}(w) = 4\cos\left(\frac{\pi w}{26}\right) + 10 \quad 0 \le w \le 52$$

where Sales(w) is the number of sales per week for $0 \le w \le 52$ and w is the time in weeks after January 1st.



a) Give the period and amplitude of the graph.

b) Find the number of sales the real estate firm makes in week 10 of the year.

c) How many weeks in a year does the model predict that sales will be less than 9?

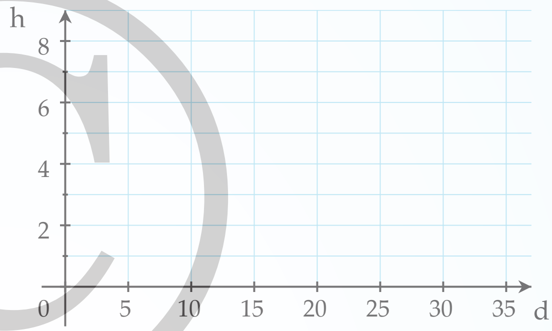
d) The sales of a real estate firm are likely to be periodic over a year. Where would you expect this model to fail to represent reality?

121. A section of a rollercoaster ride at a carnival rises and falls over a space of 35 metres. This up and down movement can be modelled by the equation

$$h = 3\sin(0.2d - 3) + 5 \text{ for } 0 \le d \le 35$$

where h is the height in metres above the ground and d the horizontal distance in metres. The angle $(0.2d - 3)$ is in radians.

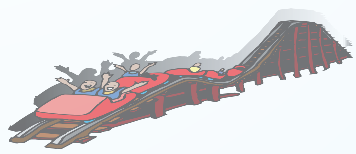
a) Graph h on the axes provided below.



b) What is the maximum height the rollercoaster reaches over this section and when does it occur?

c) What section along the course is the roller-coaster over 7 metres above ground?

d) Why is this modelling equation unlikely to model the entire ride on the rollercoaster?



Graphical Models

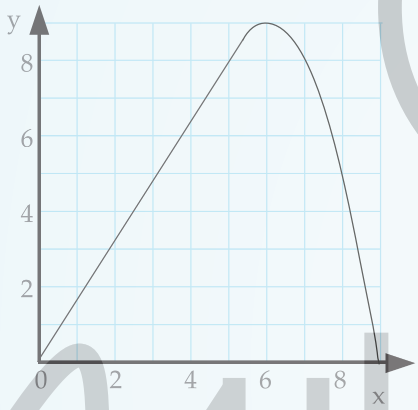
The Application of a Model

A Piecewise Function

The intention of this Achievement Standard is to use graphs to model a problem and to find solutions to that problem.

Often the appropriate model changes for different values of x.

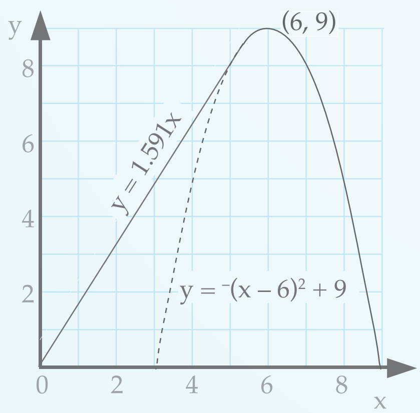
For example, the situation may be best modelled by a straight line until a certain point and then by an inverted parabola.



In this situation we specify the model for different sections of the domain or x values. This piece by piece definition is called a piecewise function.

From 0 to x = 5.5 the graph is modelled as a straight line and after this by the inverted parabola.

$$f(x) = \begin{cases} 1.591x & 0 \le x \le 5.5 \\ ^{-}(x-6)^2 + 9 & x > 5.5 \end{cases}$$



The Intersection of Two Graphs

In modelling situations we often need to find where two graphs intersect.

In this Achievement Standard it is sufficient to draw the graphs and read off the point of intersection.

It is also acceptable for this intersection to be found by algebra or by using a graphics calculator.

For example, with the piecewise function

$$f(x) = \begin{cases} 1.591x \\ ^{-}(x-6)^2 + 9 \end{cases}$$

we could confirm they intersect at x = 5.5 by making a table of values (or reading them off the graph if that is possible).

x value	y = 1.591x	$y = ^{-}(x-6)^2 + 9$
0	**0.00**	$^{-}27$
1.0	**1.59**	$^{-}16$
2.0	**3.18**	$^{-}7$
3.0	**4.77**	0
4.0	**6.36**	5
5.0	**7.96**	8
5.5	**8.75**	**8.75**
6.0	9.55	**9**
7.0	11.14	**8**
8.0	12.73	**5**
9.0	14.32	**0**

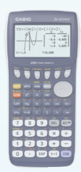
Alternatively, we could draw both graphs on our graphics calculator and find the point of intersection.

Using your TI-84 Plus after drawing the graphs select intersection with

2nd   Calc TRACE   Intersect 5

You will need to select the curves with the cursor keys and the best guess is found by moving close to each solution you need.

The solutions are (4.9, 7.8) and (5.5, 8.75).



Using your Casio 9750GII after drawing the graphs select intersection with

SHIFT   G-Solv F5   ISCT F5

The first solution is (4.9, 7.8) and the right arrow brings up the second solution (5.5, 8.75).

135. The percentage of skin cells damaged when exposed to an oxidising chemical is given by

$$D(t) = \log_b(t + 1),$$

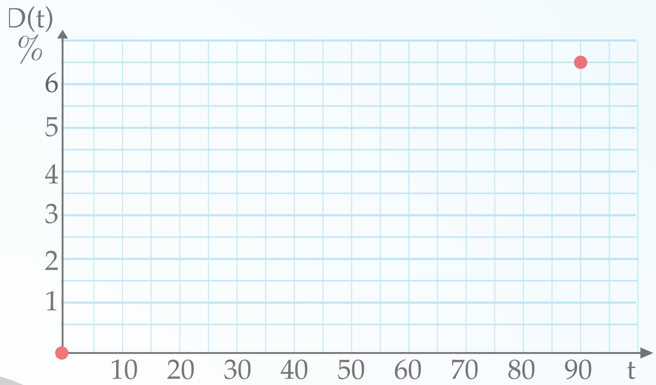
or $D(t) = k\sqrt{t}$

where t is in minutes ($t \geq 0$) and D(t) is the percentage damaged. The variables b and k are unknown. It is known that after 90 minutes, 6.5% of the skin cells are damaged.

a) Demonstrate that the base b for the logarithmic function would have to be approximately 2.

b) Calculate the value for k for the square root function.

c) Sketch both functions on the graph making sure they pass through the known points.



d) It is known that the damage to the skin starts very quickly but the rate of damage decreases over time. On this basis which of the two models would be most appropriate? Justify your answer by comparing the damage at a significant point.

136. Air is pumped into a hot air balloon. The volume of the balloon increases but at a decreasing rate as the balloon fills. The maximum volume of the balloon is 200 litres.

The inflation of the balloon started sometime in the first 3 minutes. The known volume is

Time t minutes	Volume litres
3	38
12	200

a) The volume of the balloon can possibly be modelled by a parabola. Find the equation of this parabola.

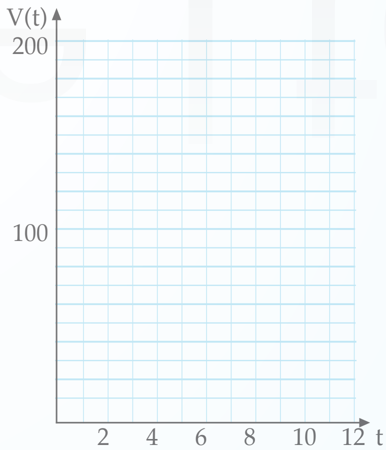
b) Another modelling equation could be

$V(t) = k\sqrt{(t - a)}$

where k is a constant and a is the time the balloon started to inflate. Calculate a and k for the known volumes, t = 3 and t = 12.

c) For each model, calculate the time at which the inflation of the balloon started?

d) Sketch both the graphs of the inflation of the balloon.

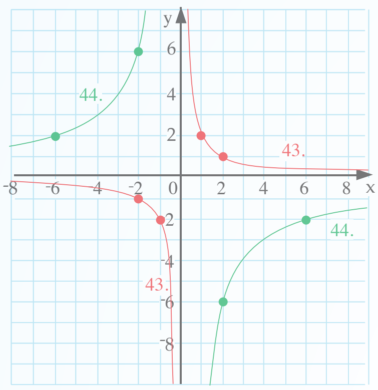


e) Which of the two modelling equations seems most appropriate?

Page 22 cont...

43. The centre of the positive hyperbola is (0, 0), the vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.

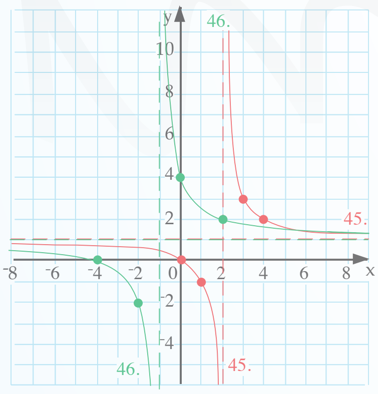
44. The centre of the negative hyperbola is (0, 0), the vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.



Page 23

45. The hyperbola has half-turn symmetry about the centre (2, 1), the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 1$.

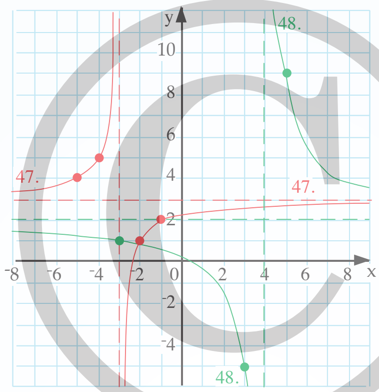
46. The hyperbola has half-turn symmetry about the centre $(^-1, 1)$, the vertical asymptote is $x = {^-1}$ and the horizontal asymptote is $y = 1$.



Page 23 cont...

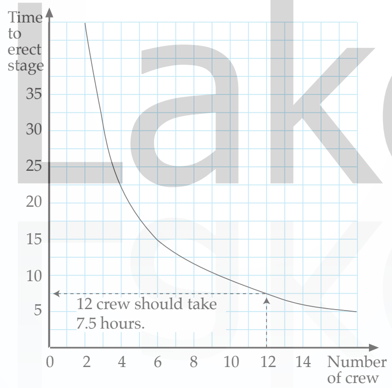
47. The hyperbola has half-turn symmetry about the point $(^-3, 3)$, the vertical asymptote is $x = {^-3}$ and the horizontal asymptote is $y = 3$.

48. The hyperbola has half-turn symmetry about the point (4, 2), the vertical asymptote is $x = 4$ and the horizontal asymptote is $y = 2$.



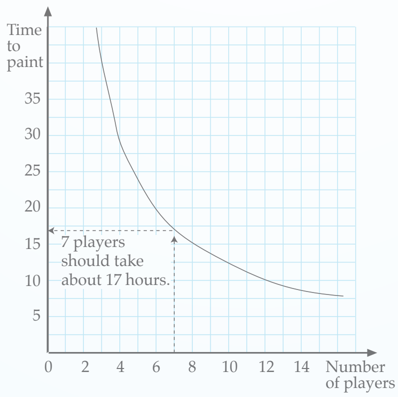
49. Equation is $\text{Time} = \dfrac{90}{\text{crew}}$

Time for 12 crew is 7.5 hours.



50. Eqn. is $\text{Time} = \dfrac{120}{\text{No. players}}$

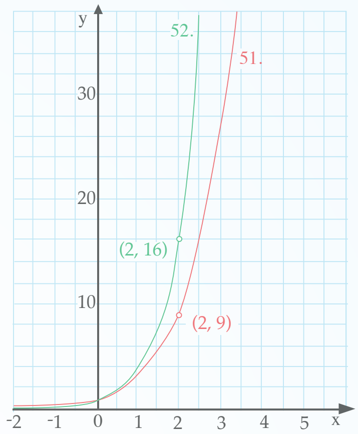
Time to paint with 7 players is 17.1 hours (1 dp).



Page 27

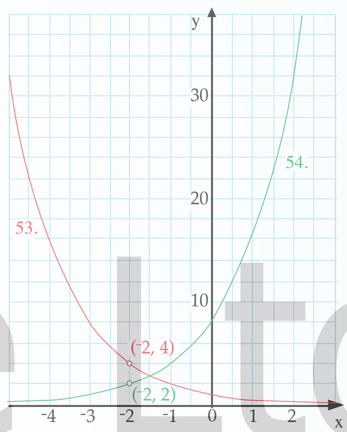
51. (2, 9)

52. (2, 16)



53. $(^-2, 4)$

54. $(^-2, 2)$



55. Translation of $y = 3^x$ across 0 up 6.

56. Translation of $y = 4^x$ across 2 up 0.